MATH2118 Lecture Notes

Further Engineering Mathematics C

Matrices

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1 Introduction

We define any rectangular arrangement of elements to be a matrix. Generally, square brackets [ ] are used to denote a matrix. For example,

$$\begin{bmatrix} x & y & z \end{bmatrix} \quad \text{is a } \textit{row matrix} \text{ or } \textit{row vector,}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{is a } \textit{column matrix} \text{ or } \textit{column vector, and}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \quad \text{is a } 3 \times 3 \text{ matrix (3 rows and 3 columns).}$$

A *square matrix* has an equal number of rows and columns. The general $m \times n$ matrix is

$$\boldsymbol{A} = \left[a_{ij}\right]_{mn} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & (a_{23}) & \cdots & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & \cdots & a_{mn} \end{bmatrix}.$$

The element a_{23} is the element in second row and third column of matrix $\boldsymbol{A}$. The dimension (or order) of a matrix is the number of rows by the number of columns.

1.1 Equality of Matrices

Two matrices are said to be *equal* if they have the same order and all corresponding elements are equal. For example,

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

both have the same order (*i.e.* 2×2 matrix) and are equal if $a_{11} = b_{11}$, $a_{12} = b_{12}$, $a_{21} = b_{21}$ and $a_{22} = b_{22}$, or $a_{ij} = b_{ij}$ for $i, j = 1, 2$.

1.2 Addition of Matrices

If $\boldsymbol{A} = [a_{ij}]_{mn}$ and $\boldsymbol{B} = [b_{ij}]_{mn}$ (both matrices must have the same order) then

$$\boldsymbol{A} + \boldsymbol{B} = \left[a_{ij} + b_{ij}\right]_{mn} = \left[b_{ij} + a_{ij}\right]_{mn} = \boldsymbol{B} + \boldsymbol{A}.$$

That is, matrices commute under addition,

$$\boxed{\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A} \qquad \text{and} \qquad \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}.}$$

For example, if $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, then

$$\boldsymbol{A} + \boldsymbol{B} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix} = \begin{bmatrix} b_{11} + a_{11} & b_{12} + a_{12} \\ b_{21} + a_{21} & b_{22} + a_{22} \end{bmatrix} = \boldsymbol{B} + \boldsymbol{A}.$$

1.3 Scalar Multiplication

If $\boldsymbol{A}$ is a matrix and k is a scalar (real or complex number) then

$$k\boldsymbol{A} = k\left[a_{ij}\right]_{mn} = \left[ka_{ij}\right]_{mn}.$$

Thus, $\boldsymbol{A} - \boldsymbol{B} = \boldsymbol{A} + (-1)\boldsymbol{B}$. For example, if $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, then

$$k\boldsymbol{A} = \begin{bmatrix} ka_{11} & ka_{12} \\ ka_{21} & ka_{22} \end{bmatrix}.$$

1.4 Matrix Multiplication

If $\boldsymbol{A} = \left[a_{ij}\right]_{mn}$ and $\boldsymbol{B} = \left[b_{jk}\right]_{rs}$, then the product $\boldsymbol{C} = \boldsymbol{AB}$ is defined only if $n = r$. That is, the number of columns in $\boldsymbol{A}$ equals the number of rows in $\boldsymbol{B}$. The order of matrix $\boldsymbol{C}$ is $m \times s$, *i.e.* $\left[a_{ij}\right]_{mn}\left[b_{jk}\right]_{rs} = \left[c_{ik}\right]_{ms}$, where

$$\begin{aligned} c_{ik} &= \sum_{j=1}^{n} a_{ij} b_{jk} \\ &= a_{i1}b_{1k} + a_{i2}b_{2k} + a_{i3}b_{3k} + \cdots + a_{in}b_{nk} \\ &= \textit{dot product} \text{ of the } i\text{th row of } \boldsymbol{A} \text{ with the } k\text{th column of } \boldsymbol{B}. \end{aligned}$$

In general, matrices are non-commutative under multiplication,

$$\boxed{\boldsymbol{AB} \neq \boldsymbol{BA}.}$$

If two matrices are not conformable (not able to be multiplied), then their product is undefined.

■ **EXAMPLE**

If $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$, find all solutions $\boldsymbol{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ of $\boldsymbol{Ax} = -3\boldsymbol{x}$.

SOLUTION

$$\boldsymbol{Ax} = -3\boldsymbol{x} \qquad \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = -3\begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

Expanding this system of equations gives

$$\begin{aligned} a + 2b &= -3a & \Rightarrow\ & 2a + b = 0 \\ 2a - b + 2c &= -3b & \Rightarrow\ & a + b + c = 0 \\ 2b + c &= -3c & \Rightarrow\ & b + 2c = 0 \end{aligned}$$

The first and third equations give $a = c = -\frac{1}{2}b$, and inserting into the second equation gives $-\frac{1}{2}b + b - \frac{1}{2}b = 0$, which is true for all real values of b. Hence,

$$\boldsymbol{x} = b\begin{bmatrix} -\frac{1}{2} \\ 1 \\ -\frac{1}{2} \end{bmatrix} = -\frac{b}{2}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

NOTE:

If all operations are defined, then

$$\begin{aligned} \boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) &= \boldsymbol{AB} + \boldsymbol{AC}, \\ (\boldsymbol{B} + \boldsymbol{C})\boldsymbol{A} &= \boldsymbol{BA} + \boldsymbol{CA}, \\ \boldsymbol{A}(\boldsymbol{BC}) &= (\boldsymbol{AB})\boldsymbol{C}. \end{aligned}$$

Since $\boldsymbol{AB} \neq \boldsymbol{BA}$, in general, we have

$$\begin{aligned}(\boldsymbol{A}-\boldsymbol{B})(\boldsymbol{A}+\boldsymbol{B}) &= \boldsymbol{A}^2+\boldsymbol{AB}-\boldsymbol{BA}-\boldsymbol{B}^2\\ &\neq \boldsymbol{A}^2-\boldsymbol{B}^2,\\ (\boldsymbol{A}+\boldsymbol{B})^2 &= (\boldsymbol{A}+\boldsymbol{B})(\boldsymbol{A}+\boldsymbol{B})\\ &= \boldsymbol{A}^2+\boldsymbol{AB}+\boldsymbol{BA}+\boldsymbol{B}^2\\ &\neq \boldsymbol{A}^2+2\boldsymbol{AB}+\boldsymbol{B}^2,\\ (\boldsymbol{AB})^2 &= \boldsymbol{ABAB}\\ &\neq \boldsymbol{A}^2\boldsymbol{B}^2.\end{aligned}$$

1.5 Tranpose of a Matrix

If $\boldsymbol{A}$ is an $m \times n$ matrix $\left[a_{ij}\right]_{mn}$, then the *transpose* of $\boldsymbol{A}$, denoted by $\boldsymbol{A}^T$, is the $n \times m$ matrix $\left[a_{ji}\right]_{nm}$ obtained by <u>interchanging</u> the rows and columns of $\boldsymbol{A}$.

For example, if $\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, then

$$\boldsymbol{A}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

The ith row of $\boldsymbol{A}^T$ is the ith column of $\boldsymbol{A}$, and the jth column of $\boldsymbol{A}^T$ is the jth row of $\boldsymbol{A}$.

<u>NOTE</u>:

$(\boldsymbol{A}+\boldsymbol{B})^T = \boldsymbol{A}^T+\boldsymbol{B}^T$ if $\boldsymbol{A}$ and $\boldsymbol{B}$ have the <u>same order</u>,

$\left(\boldsymbol{A}^T\right)^T = \boldsymbol{A}$,

$(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$ provided $\boldsymbol{AB}$ is defined ($\boldsymbol{A}$ and $\boldsymbol{B}$ are conformable).

EXAMPLE

Given $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}$ and $\boldsymbol{C} = \begin{bmatrix} 5 & 1 \\ 1 & 3 \end{bmatrix}$, show that

(a) $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC}+\boldsymbol{BC}$,

(b) $(\boldsymbol{AB})\boldsymbol{C} = \boldsymbol{A}(\boldsymbol{BC})$,

(c) $(\boldsymbol{AB})^T = \boldsymbol{B}^T\boldsymbol{A}^T$.

SOLUTION

$$\boldsymbol{A}+\boldsymbol{B} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix},$$

$$(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C} = \begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 30 & 20 \end{bmatrix},$$

$$\boldsymbol{AC} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 7 & 7 \\ 19 & 15 \end{bmatrix},$$

$$\boldsymbol{BC} = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 11 & 5 \end{bmatrix},$$

$$\Rightarrow \ \boldsymbol{AC}+\boldsymbol{BC} = \begin{bmatrix} 7 & 7 \\ 19 & 15 \end{bmatrix} + \begin{bmatrix} -1 & -3 \\ 11 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 30 & 20 \end{bmatrix}$$
$$= (\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C}.$$

$$\boldsymbol{AB} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 8 & 1 \end{bmatrix},$$

$$(\boldsymbol{AB})\boldsymbol{C} = \begin{bmatrix} 4 & 1 \\ 8 & 1 \end{bmatrix}\begin{bmatrix} 5 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 21 & 7 \\ 41 & 11 \end{bmatrix},$$

$$\Rightarrow \ \boldsymbol{A}(\boldsymbol{BC}) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -1 & -3 \\ 11 & 5 \end{bmatrix} = \begin{bmatrix} 21 & 7 \\ 41 & 11 \end{bmatrix}$$
$$= (\boldsymbol{AB})\boldsymbol{C}.$$

$$(\boldsymbol{AB})^T = \begin{bmatrix} 4 & 1 \\ 8 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & 8 \\ 1 & 1 \end{bmatrix},$$

$$\Rightarrow \ \boldsymbol{B}^T\boldsymbol{A}^T = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}^T\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 1 & 1 \end{bmatrix}$$
$$= (\boldsymbol{AB})^T.$$

2 Some Special Matrices

2.1 Zero Matrix

A matrix of any order contains only zero elements. For example,

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & \cdots & \ddots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \ddots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}.$$

Note that

$$\boxed{\boldsymbol{A} + \mathbf{0} = \mathbf{0} + \boldsymbol{A} = \boldsymbol{A} \quad \text{(identity under matrix addition)}}$$

and

$$\boxed{\boldsymbol{A}\mathbf{0} = \mathbf{0}\boldsymbol{A} = \mathbf{0}.}$$

Unlike ordinary algebra, $\boldsymbol{AB} = \mathbf{0}$ does not imply that $\boldsymbol{A} = \mathbf{0}$ or $\boldsymbol{B} = \mathbf{0}$.

EXAMPLE

Given $\boldsymbol{A} = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$, we have

$$\begin{aligned} \boldsymbol{AB} &= \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \\ &= \mathbf{0}. \end{aligned}$$

Even though $\boldsymbol{A} \neq \mathbf{0}$ and $\boldsymbol{B} \neq \mathbf{0}$, we obtain $\boldsymbol{AB} = \mathbf{0}$.

2.2 Unit (Identity) Matrix

The unit matrix $\boldsymbol{I}_n$ (or $\boldsymbol{I}$ when the order is known) is a square matrix of order $n \times n$ with entries of 1's down the main diagonal and 0's everywhere else. For example,

$$\boldsymbol{I}_n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \ddots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

For any square matrix $\boldsymbol{A}$,

$$\boxed{\boldsymbol{AI} = \boldsymbol{IA} = \boldsymbol{A}.}$$

2.3 Diagonal Matrices

A square matrix in which all off-diagonal elements are zero. If both $\boldsymbol{A}$ and $\boldsymbol{B}$ are diagonal matrices of the same order, then $\boldsymbol{A} + \boldsymbol{B}$ and $\boldsymbol{AB}$ are also diagonal, and $\boldsymbol{AB} = \boldsymbol{BA}$.
For example,

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \quad \text{where } a_{11},\, a_{22},\, a_{33} \neq 0.$$

2.4 Triangular Matrices

A *lower triangular matrix* is a square matrix where all elements above the main diagonal are zero, such as

$$\boldsymbol{L} = \begin{bmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix}.$$

An *upper triangular matrix* is a square matrix where all elements below the main diagonal are zero. For example,

$$\boldsymbol{U} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}.$$

2.5 Symmetric and Antisymmetric Matrices

A square matrix $\boldsymbol{A}$ is *symmetric* if $\boldsymbol{A}^T = \boldsymbol{A}$, *i.e.* $a_{ij} = a_{ji}$, displaying symmetry about the main diagonal. Matrix $\boldsymbol{A}$ is *anti-symmetric* (or *skew-symmetric*) if $\boldsymbol{A}^T = -\boldsymbol{A}$, *i.e.* $a_{ij} = -a_{ji}$, displaying anti-symmetry about the main diagonal (the diagonal will consist solely of zero elements).

EXAMPLE

Matrix $\boldsymbol{A}$ is anti-symmetric, since

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 2 \\ -1 & 0 & 3 \\ -2 & -3 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{A}^T = \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix} = -\boldsymbol{A}.$$

2.6 Orthogonal Matrices

A square matrix $\boldsymbol{P}$ is said to be *orthogonal* if

$$\boxed{\boldsymbol{P}\boldsymbol{P}^T = \boldsymbol{P}^T\boldsymbol{P} = \boldsymbol{I}.}$$

The columns (rows) of an orthogonal matrix are mutually orthogonal unit vectors. For example,

$$\boldsymbol{P} = \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{bmatrix}.$$

2.7 Invertible Matrices

A square matrix $\boldsymbol{A}$ is said to be *invertible* (or *non-singular*) if there exists a square matrix $\boldsymbol{B}$ such that

$$\boxed{\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{A} = \boldsymbol{I}.}$$

The matrix $\boldsymbol{B}$ is then called the *inverse* of $\boldsymbol{A}$, denoted by $\boldsymbol{A}^{-1}$, and its inverse is *unique*.

EXAMPLE

Consider matrix $\boldsymbol{A} = \begin{bmatrix} 2 & -5 \\ -1 & -3 \end{bmatrix}$. The inverse of $\boldsymbol{A}$ is

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 3/11 & -5/11 \\ -1/11 & -2/11 \end{bmatrix},$$

since

$$\begin{aligned} \boldsymbol{A}\boldsymbol{A}^{-1} &= \begin{bmatrix} 2 & -5 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 3/11 & -5/11 \\ -1/11 & -2/11 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \boldsymbol{I}. \end{aligned}$$

REMARKS:

Note that not every square matrix has an inverse. If $|\boldsymbol{A}| = 0$ then $\boldsymbol{A}^{-1}$ cannot exist, since

$$|\boldsymbol{A}\boldsymbol{B}| = |\boldsymbol{A}| \times |\boldsymbol{B}|,$$

and if $|\boldsymbol{A}| = 0$, then $|\boldsymbol{A}\boldsymbol{B}| = 0$, while $|\boldsymbol{I}| = 1$. Hence, it is impossible for $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{I}$, because $|\boldsymbol{A}\boldsymbol{B}| \neq |\boldsymbol{I}|$. If both $\boldsymbol{A}$ and $\boldsymbol{B}$ are non-singular matrices of the same order, then

$$\begin{aligned} \left(\boldsymbol{A}^{-1}\right)^{-1} &= \boldsymbol{A}, \\ \left(\boldsymbol{A}^{T}\right)^{-1} &= \left(\boldsymbol{A}^{-1}\right)^{T}, \\ (\boldsymbol{A}\boldsymbol{B})^{-1} &= \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}. \end{aligned}$$

2.8 Diagonalisation of Matrix

Square matrix $\boldsymbol{P}$ diagonalises matrix $\boldsymbol{A}$ if

$$\boxed{\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} = \boldsymbol{I}.}$$

3 Gaussian Elimination

Consider the following system of equations:

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0, \\ 2x_1 + 2x_2 + 3x_3 &= 3, \\ -x_1 - 3x_2 &= 2. \end{aligned}$$

The coefficients of the unknowns (the symbols used for the unknowns are not important) together with the right-hand-side constants are all that is needed when determining the solution. Rewrite the system of equations in an *augmented matrix* form shown below,

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 2 & 2 & 3 & 3 \\ -1 & -3 & 0 & 2 \end{array}\right] \begin{array}{l} \mathrm{R}_1 \\ \mathrm{R}_2 \\ \mathrm{R}_3 \end{array}$$

STEP 1:
Eliminate x_1 from R_2 and R_3. We define *multipliers*,

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{2}{1} = 2 \quad \text{and} \quad m_{31} = \frac{a_{31}}{a_{11}} = \frac{-1}{1} = -1,$$

and substract $m_{21}\mathrm{R}_1$ from R_2, and substract $m_{31}\mathrm{R}_1$ from R_3:

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & -1 & 1 & 2 \end{array}\right] \begin{array}{l} \\ \mathrm{R}_2 - 2\mathrm{R}_1 \\ \mathrm{R}_3 + \mathrm{R}_1 \end{array}$$

STEP 2:
Eliminate x_2 from R_3 by substracting $\frac{1}{2}\mathrm{R}_2$ from R_3:

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}\right] \begin{array}{l} \\ \\ \mathrm{R}_3 - \frac{1}{2}\mathrm{R}_2 \end{array}$$

All elements below the main diagonal are zero, this matrix is said to be in *row-echelon* form.

STEP 3:
Work backwards from the last to first equation:

$$\begin{aligned} \tfrac{1}{2}x_3 &= \tfrac{1}{2} & &\Rightarrow\ x_3 = 1 \\ -2x_2 + x_3 &= 3 & &\Rightarrow\ x_2 = -1 \\ x_1 + 2x_2 + x_3 &= 0 & &\Rightarrow\ x_1 = 1 \end{aligned}$$

Gaussian elimination is the most efficient means of solving a system of linear equations.

EXAMPLE

Determine the solutions (if they exist) of the following system of equations,

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 2, \\ 4x_1 + 5x_2 + 6x_3 &= 8, \\ 7x_1 + 8x_2 + 9x_3 &= 13. \end{aligned}$$

SOLUTION

Write this system of equations in an augmented matrix form,

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 4 & 5 & 6 & 8 \\ 7 & 8 & 9 & 13 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & -1 \end{array}\right] \begin{array}{l} \\ \mathrm{R}_2 - 4\mathrm{R}_1 \\ \mathrm{R}_3 - 7\mathrm{R}_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right] \begin{array}{l} \\ -\frac{1}{3}\mathrm{R}_2 \\ \mathrm{R}_3 - 2\mathrm{R}_2 \end{array}$$

The last row of this matrix implies $0x_1 + 0x_2 + 0x_3 = -1$, which is clearly impossible. Hence, this system has no solution; such systems of equations are said to be *inconsistent*.

EXAMPLE

Determine the solutions (if they exist) of the following system of equations,

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 2, \\ 4x_1 + 5x_2 + 6x_3 &= 8, \\ 7x_1 + 8x_2 + 9x_3 &= 14. \end{aligned}$$

SOLUTION

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 4 & 5 & 6 & 8 \\ 7 & 8 & 9 & 14 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & -12 & 0 \end{array}\right] \begin{array}{l} \\ \mathrm{R}_2 - 4\mathrm{R}_1 \\ \mathrm{R}_3 - 7\mathrm{R}_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \\ -\frac{1}{3}\mathrm{R}_2 \\ \mathrm{R}_3 - 2\mathrm{R}_2 \end{array}$$

The last row of zeros indicates that the last equation in the system is redundant. Working backward from R_2 to R_1, where x_3 can take any value, we have

$$\begin{aligned} x_2 + 2x_3 &= 0 &\Rightarrow x_2 &= -2x_3 \\ x_1 + 2x_2 + 3x_3 &= 2 &\Rightarrow x_1 &= 2 - 2x_2 - 3x_3 = 2 + x_3 \end{aligned}$$

For instance, if $x_3 = \alpha$ (any real value), the general solution is

$$\begin{aligned} x_1 &= 2 + \alpha \\ x_2 &= -2\alpha \\ x_3 &= \alpha \end{aligned} \quad \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix},$$

or

$$(x_1, x_2, x_3) = (2, 0, 0) + \alpha(1, -2, 1); \quad \alpha \in R.$$

4 Inverse of a Matrix

If the matrix $\boldsymbol{A}$ is non-singular, *i.e.* $|\boldsymbol{A}| \neq 0$, then its inverse $\boldsymbol{A}^{-1}$ may be found in the following way:

(1) Write down $\boldsymbol{A}$ with the unit matrix $\boldsymbol{I}$ in an augmented matrix form, *i.e.* $\left[\boldsymbol{A} | \boldsymbol{I}\right]$.

(2) Perform a sequence of elementary row operations on $\left[\boldsymbol{A} \,|\, \boldsymbol{I}\right]$ until $\boldsymbol{A}$ becomes $\boldsymbol{I}$.

(3) By then matrix $\boldsymbol{I}$ will have been converted into the inverse matrix $\boldsymbol{A}^{-1}$, *i.e.* $\left[\boldsymbol{I} \,|\, \boldsymbol{A}^{-1}\right]$.

NOTE:

For a 2×2 matrix, $\boldsymbol{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, its matrix inverse can be found using the following formula

$$\boldsymbol{A}^{-1} = \frac{1}{|\boldsymbol{A}|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{where } |\boldsymbol{A}| = ad - bc.$$

This is only for a 2×2 matrix.

EXAMPLE

If $\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$, find $\boldsymbol{A}^{-1}$ using elementary row operations.

SOLUTION

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} \\ \mathrm{R}_2 - \mathrm{R}_1 \\ \\ \end{matrix}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right] \begin{matrix} \mathrm{R}_1 + \mathrm{R}_2 \\ \\ \mathrm{R}_3 + \mathrm{R}_2 \end{matrix}$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 1 & 1 & -1 \\ 0 & -2 & 0 & -1 & 1 & -1 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right] \begin{matrix} 2\mathrm{R}_1 - \mathrm{R}_3 \\ 2\mathrm{R}_2 - \mathrm{R}_3 \\ \\ \end{matrix}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right] \begin{matrix} \frac{1}{2}\mathrm{R}_1 \\ -\frac{1}{2}\mathrm{R}_2 \\ \frac{1}{2}\mathrm{R}_3 \end{matrix}$$

Inverse of $\boldsymbol{A}$:

$$\boldsymbol{A}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}.$$

<u>Check</u>: Show $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$ holds for $\boldsymbol{A}$:

$$\begin{aligned} \boldsymbol{A}\boldsymbol{A}^{-1} &= \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \boldsymbol{I}. \end{aligned}$$

5 Determinant of a Matrix

To each square matrix there is associated a number called the *determinant* of a matrix. For example, for $n \times n$ matrix $\boldsymbol{A}$,

$$\det \boldsymbol{A} = \left|\boldsymbol{A}\right| = \begin{vmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \cdots & \vdots \\ \vdots & \vdots & \cdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{vmatrix}.$$

For a 2×2 matrix, $\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, the 2×2 determinant is

$$\boxed{\det\boldsymbol{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}\,.}$$

A 3×3 determinant is defined in terms of 2×2 determinants,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{\text{minor of } a_{11}} - a_{12}\underbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}_{\text{minor of } a_{12}} + a_{13}\underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{\text{minor of } a_{13}}.$$

The minor of a_{11} is obtained from the original determinant by deleting the row and column which contains a_1, etc. Note the sequence of signs $+ - +$ associated with the coefficients a_{11}, a_{12} and a_{13}, respectively. Alternatively, we can expand the determinant across the second or third rows, or even down any of the columns.

For example, expanding down the first column gives

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{\text{minor of } a_{11}} - a_{21}\underbrace{\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}}_{\text{minor of } a_{21}} + a_{31}\underbrace{\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}}_{\text{minor of } a_{31}}.$$

Note the following pattern of signs when expanding a 3×3 determinant,

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

High-order determinants can be expanded in terms of lower-order determinants in a similar manner to 3×3 determinants where the pattern of signs extends naturally as follows,

$$\begin{matrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{matrix}$$

5.1 Properties of Determinants

For square matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and a scalar k:

(1) If $\boldsymbol{A} = \boldsymbol{B}\boldsymbol{C}$ then $|\boldsymbol{A}| = |\boldsymbol{B}||\boldsymbol{C}|$.

However, if $\boldsymbol{A} = \boldsymbol{B} + \boldsymbol{C}$, then $|\boldsymbol{A}| \neq |\boldsymbol{B}| + |\boldsymbol{C}|$.

(2) If two columns (rows) of A are identical then $|\boldsymbol{A}| = 0$.

(3) If one column (row) of A is the zero vector, then $|\boldsymbol{A}| = 0$.

For example,

$$\begin{vmatrix} 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - 0 \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + 0 \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = 0.$$

(4) Interchanging two columns (rows) of a determinant changes the sign of the determinant.

(5) The determinant is unaltered if a scalar multiple of one row (column) is added to another row (column).

(6) Interchanging the rows and columns of a determinant does not alter its value,

$$\left|\boldsymbol{A}^T\right| = |\boldsymbol{A}|.$$

(7) If $\overline{\boldsymbol{A}}$ is the matrix obtained from matrix $\boldsymbol{A}$ by replacing the jth column (row), vector $\boldsymbol{A}_j$, by $k\boldsymbol{A}_j$, then $|\bar{\boldsymbol{A}}| = k\,|\boldsymbol{A}|$. For example,

$$\begin{aligned} \begin{vmatrix} ka_{11} & ka_{12} \\ a_{21} & a_{22} \end{vmatrix} &= ka_{11}a_{22} - ka_{12}a_{21} \\ &= k(a_{11}a_{22} - a_{12}a_{21}) \\ &= k \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}. \end{aligned}$$

EXAMPLE

$$\begin{vmatrix} 2 & -3 \\ -4 & 3 \end{vmatrix} = (2 \times 3) - (-4 \times -3)$$
$$= 6 - 12$$
$$= -6.$$

EXAMPLE

Expanding along the last column gives

$$\begin{vmatrix} -1 & 3 & 1 \\ 2 & 5 & 0 \\ 3 & 2 & -1 \end{vmatrix} = 1 \begin{vmatrix} 2 & 5 \\ 3 & 2 \end{vmatrix} + 0 \begin{vmatrix} -1 & 3 \\ 3 & 2 \end{vmatrix} - 1 \begin{vmatrix} -1 & 3 \\ 2 & 5 \end{vmatrix}$$
$$= (4 - 15) + 0 - (-5 - 6)$$
$$= 0.$$

6 Eigenvalues and Eigenvectors

Consider any linear transformation $\boldsymbol{A}$, which operates on the column vector $\boldsymbol{v}$. If the condition

$$\boxed{\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v}}$$

is satisfied for λ (a scalar), then we have a situation where the transformation $\boldsymbol{A}$ *does not change* the direction of $\boldsymbol{v}$. The vector $\boldsymbol{V}$ is called an *eigenvector* of $\boldsymbol{A}$, and the scalar λ is called an *eigenvalue* of $\boldsymbol{A}$; λ may be zero, real or complex value.

Note that if $\boldsymbol{v}$ is an eigenvector, then so is $k\boldsymbol{v}$ for some constant k. Thus, an eigenvector is simply a vector which maps on a scalar multiple of itself, while eigenvalue gives a measure of how the eigenvector is "stretched". Note that

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v} = \lambda \boldsymbol{I}\boldsymbol{v},$$

where $\boldsymbol{I}$ is the identity matrix (same dimension as $\boldsymbol{A}$), then

$$\big(\boldsymbol{A} - \lambda \boldsymbol{I}\big)\boldsymbol{v} = \boldsymbol{0} \quad \text{and} \quad |\boldsymbol{A} - \lambda \boldsymbol{I}||\boldsymbol{v}| = 0.$$

Since $\boldsymbol{v} \neq \boldsymbol{0}$, so $|\boldsymbol{v}| \neq 0$, it follows that

$$\boxed{\det\big(\boldsymbol{A} - \lambda \boldsymbol{I}\big) = |\boldsymbol{A} - \lambda \boldsymbol{I}| = 0,}$$

which is the *characteristic equation of the matrix* $\boldsymbol{A}$ (nth degree polynomial for $n \times n$ matrix $\boldsymbol{A}$). Hence, there are n real or complex eigenvalues (roots of its characteristic equation) for an $n \times n$ matrix including repeated roots.

EXAMPLE

Determine the eigenvalues and eigenvectors of $\boldsymbol{A} = \begin{bmatrix} 1 & -2 & 7 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$.

<u>SOLUTION</u>

Computing $\boldsymbol{A} - \lambda \boldsymbol{I}$:

$$\boldsymbol{A} - \lambda \boldsymbol{I} = \begin{bmatrix} 1 & -2 & 7 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1-\lambda & -2 & 7 \\ 0 & -1-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{bmatrix}.$$

Characteristic equation:

$$\big|\boldsymbol{A} - \lambda \boldsymbol{I}\big| = \begin{vmatrix} 1-\lambda & -2 & 7 \\ 0 & -1-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$
$$= 0 - 0 + (2-\lambda) \begin{vmatrix} 1-\lambda & -2 \\ 0 & -1-\lambda \end{vmatrix}$$
$$= (2-\lambda)\big[(1-\lambda)(-1-\lambda) - 0\big]$$
$$= -(2-\lambda)(1-\lambda)(1+\lambda).$$

Setting $|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0$ gives the eigenvalues $\lambda = 2,\ 1,\ -1$.

<u>Eigenvector for $\lambda = -1$:</u>

Find all non-zero vectors, $\boldsymbol{v} = [a \;\; b \;\; c]^T$, which satisfy

$$\boldsymbol{A}\boldsymbol{v} = \lambda \boldsymbol{v} \quad \text{or} \quad \big(\boldsymbol{A} - \lambda \boldsymbol{I}\big)\boldsymbol{v} = \boldsymbol{0}.$$

For $\lambda = -1$,

$$\begin{bmatrix} 1-\lambda & -2 & 7 \\ 0 & -1-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 & -2 & 7 \\ 0 & 0 & 3 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Expanding this matrix equation gives

$$2a - 2b + 7c = 0,$$
$$3c = 0,$$
$$3c = 0.$$

Thus, $c = 0$ and $a = b$,

$$\boldsymbol{v} = a\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

A possible eigenvector is $(1, 1, 0)$.

Eigenvector for $\lambda = 1$:

$$\begin{bmatrix} 1-\lambda & -2 & 7 \\ 0 & -1-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 & -2 & 7 \\ 0 & -2 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Expanding this matrix equation gives

$$\begin{aligned} -2b + 7c &= 0, \\ -2b + 3c &= 0, \\ c &= 0. \end{aligned}$$

Thus, for $b = c = 0$, we have

$$\boldsymbol{v} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

and a possible eigenvector is $(1, 0, 0)$.

Eigenvector for $\lambda = 2$:

$$\begin{bmatrix} 1-\lambda & -2 & 7 \\ 0 & -1-\lambda & 3 \\ 0 & 0 & 2-\lambda \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -1 & -2 & 7 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This gives

$$\begin{aligned} -a - 2b + 7c &= 0, \\ -3b + 3c &= 0, \\ 0 &= 0. \end{aligned}$$

Thus, for $b = c$ and $a = -2b + 7c = 5c$,

$$\boldsymbol{v} = c\begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix},$$

and a possible eigenvector is $(5, 1, 1)$.

NOTE:

If we construct a non-singular matrix, $\boldsymbol{P}$, whose columns are eigenvectors ($\boldsymbol{x}_i$), corresponding to the distinct eigenvalues, λ_i, of matrix $\boldsymbol{A}$, respectively,

$$\boldsymbol{P} = \Big[\, \boldsymbol{x}_1,\ \boldsymbol{x}_2,\ \ldots,\ \boldsymbol{x}_n \,\Big],$$

then $\boldsymbol{P}^{-1}\boldsymbol{AP}$ is a diagonal matrix, $\boldsymbol{D}$, consists of λ_i values only,

$$\boldsymbol{P}^{-1}\boldsymbol{AP} = \boldsymbol{D} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & \ldots & \ldots & \ldots \\ \ldots & \ldots & \lambda_i & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ddots & \ldots \\ 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix}.$$

The λ_i appear in the same order along the diagonal as the order of the eigenvector $\boldsymbol{x}_i$ in the column of $\boldsymbol{P}$. Note that $\boldsymbol{P}$ is singular if $\boldsymbol{A}$ does not have distinct eigenvalues.

EXAMPLE

The eigenvectors of the last example are

$$\boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{x}_3 = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}$$

correspond to $\lambda_1 = -1$, $\lambda_2 = 1$ and $\lambda_3 = 2$, respectively. Show that $\boldsymbol{P}^{-1}\boldsymbol{AP} = \boldsymbol{D}$ is valid for the matrix $\boldsymbol{P}$ given below,

$$\boldsymbol{P} = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \Big[\, \boldsymbol{x}_1,\ \boldsymbol{x}_2,\ \boldsymbol{x}_3 \,\Big].$$

SOLUTION

Recalling that $\boldsymbol{A} = \begin{bmatrix} 1 & -2 & 7 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$. Computing inverse of $\boldsymbol{P}$ as follows,

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 5 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} R_2 \to R_1 \\ R_1 \to R_2 \\ \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 4 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \mathrm{R}_2 - \mathrm{R}_1$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & -1 & -4 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \mathrm{R}_1 - \mathrm{R}_3 \\ \mathrm{R}_2 - 4\mathrm{R}_3 \\ \end{array}$$

Thus, the inverse is $\boldsymbol{P}^{-1} = \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & -4 \\ 0 & 0 & 1 \end{bmatrix}$.

<u>Check</u>: Verifying $\boldsymbol{P}\boldsymbol{P}^{-1} = \boldsymbol{I}$,

$$\boldsymbol{P}\boldsymbol{P}^{-1} = \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & -4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \boldsymbol{I}.$$

Hence,

$$\begin{aligned} \boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P} &= \begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 7 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 & 1 \\ 1 & -1 & -4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 5 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \\ &= \boldsymbol{D}. \end{aligned}$$

EXAMPLE

Find the eigenvalues of $\boldsymbol{A} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$.

SOLUTION

Evaluating $|\boldsymbol{A} - \lambda \boldsymbol{I}|$ first,

$$\begin{aligned} |\boldsymbol{A} - \lambda \boldsymbol{I}| &= \begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} \\ &= (1-\lambda)(-2-\lambda) - 4 \\ &= -2 - \lambda + 2\lambda + \lambda^2 - 4 \\ &= \lambda^2 + \lambda - 6 \\ &= (\lambda + 3)(\lambda - 2). \end{aligned}$$

Setting $|\boldsymbol{A} - \lambda \boldsymbol{I}| = 0$ gives the eigenvalues of $\boldsymbol{A}$,

$$\lambda_1 = -3 \quad \text{and} \quad \lambda_2 = 2.$$

7 Review Questions

[1] Determine the solutions of the systems of equations:

(a)

$$
\begin{aligned}
x_1 - 2x_2 + 3x_3 &= 5, \\
2x_1 + x_2 - 5x_3 &= -7, \\
4x_1 - 3x_2 + 2x_3 &= 5.
\end{aligned}
$$

(b)

$$
\begin{aligned}
2x_1 - x_2 + 7x_3 &= 18, \\
x_1 + x_2 + x_3 &= 3, \\
5x_1 + 2x_2 + 3x_3 &= 6.
\end{aligned}
$$

[2] Find the general solution of the system of equations:

(a)

$$
\begin{aligned}
x_1 + x_2 + x_3 + 2x_4 - x_5 &= 0, \\
2x_1 - x_2 - x_3 + x_4 + 2x_5 &= 0, \\
x_1 + 3x_2 - 2x_3 + x_4 + x_5 &= 0.
\end{aligned}
$$

(b)

$$
\begin{aligned}
x_1 - 2x_2 + x_3 - x_4 &= 0, \\
2x_1 + 4x_2 - 3x_3 &= 0, \\
3x_1 + 2x_2 + 2x_3 - x_4 &= 0.
\end{aligned}
$$

[3] If $\boldsymbol{A} = \begin{bmatrix} 3 & 2 & -2 \\ -1 & -4 & 1 \\ 2 & -4 & -1 \end{bmatrix}$, show that $\boldsymbol{A}^3 + 2\boldsymbol{A}^2 - \boldsymbol{A} - 2\boldsymbol{I} = \mathbf{0}$.

[4] Determine the inverse (if it exists) of each of the following matrices:

(a) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$;

(b) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$;

(c) $\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$.

[5] By solving

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(where $ad - bc \neq 0$) for x, y, z and w, show that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Hence, write down the inverse of $\boldsymbol{A} = \begin{bmatrix} -3 & -7 \\ 2 & 8 \end{bmatrix}$.

[6] Determine the eigenvalues and a set of three linearly independent eigenvectors for each of the following matrices $\boldsymbol{A}$:

(a) $\begin{bmatrix} 1 & -2 & 7 \\ 0 & -1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$;

(b) $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$.

8 Answers to Review Questions

[1] (a) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

(b) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 3 \end{bmatrix}$$

[2] (a) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \alpha \begin{bmatrix} -5 \\ -2 \\ -3 \\ 5 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -5 \\ 6 \\ 14 \\ 0 \\ 15 \end{bmatrix}$$

(b) $$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 1 \\ 0 \\ -4 \end{bmatrix}$$

[3] Not available.

[4] (a) $$\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

(b) No inverse exists.

(c) $$\frac{1}{4}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

[5] $$\begin{bmatrix} -4/5 & -7/10 \\ 1/5 & 3/10 \end{bmatrix}$$

[6] (a) Eigenvalues 1, -1 and 2 with the corresponding eigenvectors,

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}, \text{ respectively.}$$

(b) Eigenvalues -1, -1 and 8 with the corresponding eigenvectors,

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \text{ respectively.}$$